A New Method for Performing Digital Control System Attitude Computations Using Quaternions

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Space vehicle attitude control system performance is often limited by the computational speed of airborne digital computer hardware. The application of quaternions to digital attitude control problems involving coordinate system transformations can reduce computation time by more than 40% over the equivalent direction cosine matrix solution due to a new method of treating quaternions which allows the order of multiplication to be interchanged so as to isolate the most rapidly varying parameters. This paper presents a formulation for control equations in which the control error is expressed as a quaternion and all coordinate system transformations are performed using quaternions. The new principle of "quaternion algebra" which allows interchanging the order of multiplication is developed and used to simplify the control equations. The quaternion control equations are applied to a three-gimbal IMU example and the results are compared to an equivalent direction cosine matrix solution.

Introduction

THE problem of describing the relationship between two coordinate systems is one of the most basic concepts encountered in the field of navigation and guidance. Until recently Euler angles formed the most widely used method of describing the rotation between two coordinate systems. The use of three Euler angles to fix the attitude of a body with respect to an inertial or reference coordinate system has the advantage of being well-defined geometrically and fairly simple to visualize. The 3×3 direction cosine matrix, however, was more readily adaptable to high-speed digital computation and has replaced the Euler angle method in the solution of all but the most simple navigation problems. The direction cosine matrix approach is particularly useful to describe several successive rotations of a body with respect to a fixed reference system. A third, but infrequently applied, approach to establishing body orientation utilizes the quaternion, first devised by Hamilton. The quaternion approach makes use of Euler's Theorem which states that any real rotation of one coordinate system with respect to another may be described by a rotation through some angle about a single fixed axis. The quaternion is a compact form for representing the single fixed axis and angle referred to by Euler's Theorem. The quaternion may be handled much the same as the direction cosine matrix, in that successive rotations result in successive quaternion multiplication. The advantage of the quaternion lies in its ability to define the rotational relationship between two coordinate systems using only four numbers as opposed to the nine elements of a direction cosine matrix. This results in a similar simplification when the effect of several successive rotations is being computed. The principal impediment to use of quaternions has been that the direction cosine matrix, rather than the quaternion, is the desired end product of computation. Usually, the computation saved by using quaterion multiplication to perform successive coordinate system rotations is lost when the resultant quaternion must be put in the form of a direction cosine matrix.

When the object of the computation is to obtain the control error it is not necessary to define the direction cosine matrix explicitly. In fact, optimum control, in the least angle sense, is performed by defining the control error in terms of the single axis and angle of Euler's Theorem. This may be done by formulating the problem completely in terms of quaternions. The quaternion formulation is shown to be particularly advantageous due to a new method for treating quaternions which allows the order of multiplication to be interchanged.

A Quaternion Solution to the Attitude Problem†

The relationship between two coordinate systems having a common origin may be defined by 1) three Euler angles about predefined axes, 2) a 3×3 direction cosine matrix, 3) a single Euler axis and rotation angle. Consider the two coordinate systems, x,y,z and X,Y,Z shown in Fig. 1. Let the system x,y,z by denoted by A and X,Y,Z be denoted by B. The co-

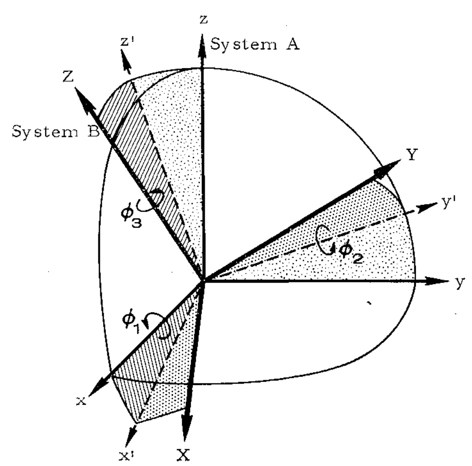

Fig. 1 Euler angle rotation.

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† The basic material presented in this section may be found in Ref. 1, where it is shown that quaternion and Euler parameter representations are equivalent.

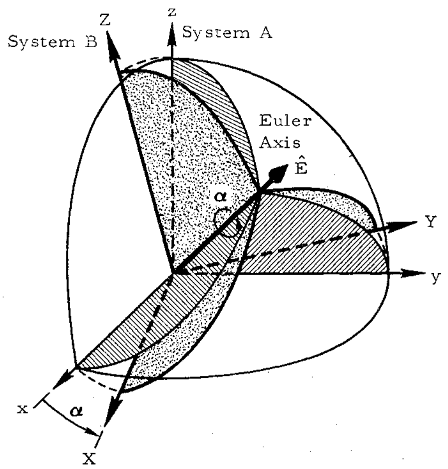

Fig. 2 Rotation about the Euler axis.

ordinate system X,Y,Z could be obtained by first rotating x,y,z through φ_1, about x, to obtain x,y',z'. The rotation φ_2 about y' produces x',y',Z and the third rotation of φ_3 about Z yields X,Y,Z. The rotation may also be described by the rotation matrix $\{R_{ij}\}$ where

$$\{R_{ij}\} \equiv [B \leftarrow A] = \begin{bmatrix} X\cdot x & X\cdot y & X\cdot z \\ Y\cdot x & Y\cdot y & Y\cdot z \\ Z\cdot x & Z\cdot y & Z\cdot z \end{bmatrix} \quad (1)$$

where

$$X\cdot x = \cos\varphi_2 \cos\varphi_3$$
$$X\cdot y = \cos\varphi_3 \sin\varphi_2 \sin\varphi_1 + \sin\varphi_3 \cos\varphi_1$$
$$X\cdot z = \sin\varphi_3 \sin\varphi_1 - \cos\varphi_3 \sin\varphi_2 \cos\varphi_1$$
$$Y\cdot x = -\sin\varphi_3 \cos\varphi_2$$
$$Y\cdot y = \cos\varphi_1 \cos\varphi_3 - \sin\varphi_3 \sin\varphi_2 \sin\varphi_1$$
$$Y\cdot z = \sin\varphi_3 \sin\varphi_2 \cos\varphi_1 + \cos\varphi_3 \sin\varphi_1$$
$$Z\cdot x = \sin\varphi_2$$
$$Z\cdot y = -\cos\varphi_2 \sin\varphi_1,\ Z\cdot z = \cos\varphi_1 \cos\varphi_2$$

The third method of describing this rotation is as a rotation α about a single axis $\hat{E}$. According Euler's Theorem, this can always be done. The relationship between $\hat{E}$, α, and $[B \leftarrow A] = \{R_{ij}\}$ is defined in a number of mathematical works[2] as

$$\hat{E} = E_x,E_y,E_z,\ E_x = (R_{23} - R_{32})/2 \sin\alpha$$
$$E_y = (R_{31} - R_{13})/2 \sin\alpha,\ E_z = (R_{12} - R_{21})/2 \sin\alpha \quad (2)$$

also

$$\cos\alpha = \tfrac{1}{2}[R_{11} + R_{22} + R_{33} - 1]$$

Note the unique property of the Euler axis: $\hat{E} = [B \leftarrow A]\hat{E}$. This is due to the fact that the rotation occurs about $\hat{E}$ and therefore does not affect the orientation of the axis $\hat{E}$.

Use of Euler's Theorem to Define Control Error

Figure 2 shows the rotation from system A to system B about the Euler axis $\hat{E}$ (unit vector). It is apparent that the rotation α is smaller than the algebraic sum of the Euler angles φ_1, φ_2, and φ_3 shown in Fig. 1, and that the angle α is the shortest angular path between the two coordinate systems. It is logical, therefore, to choose the rotation about the Euler axis to rotate from system A to system B. This may be readily accomplished by controlling the system torque vector to lie along the Euler axis. Assuming the attitude control system is capable of applying torque about the vehicle body roll, pitch, and yaw (x, y, and z) axes independently, the roll, pitch, and yaw control errors will be proportional to the x, y, and z components (or direction cosines) of the Euler axis; E_x,E_y,E_z. In addition, the control law must include the magnitude of the rotation to be performed, α. A reasonable set of control errors is the product of the Euler axis (E_x,E_y,E_z) and the Euler angle α;

$$[\epsilon_r,\epsilon_p,\epsilon_y] = [\alpha E_x,\alpha E_y,\alpha E_z]$$

Use of the Quanternion to Define Control Error

The quaternion is a "vector" with four components (q_1, q_2,q_3,q_4) which contains the information $\hat{E}$ and α and satisfies the normality condition $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$. The quaternion, $\bar{Q}_{BA}$, which relates X,Y,Z to x,y,z is written as

$$\bar{Q}_{BA} = [q_1,q_2,q_3,q_4] = \left[\cos\frac{\alpha}{2},\ E_x \sin\frac{\alpha}{2},\ E_y \sin\frac{\alpha}{2},\ E_z \sin\frac{\alpha}{2}\right] \quad (3)$$

The quaternion was developed by Hamilton in 1843 and the form of Eq. (3) was adopted because of computational simplicity in combining quaternions due to successive rotations.

For control purposes, the last three elements of the quaternion define the roll, pitch, and yaw rotational errors

$$[\epsilon_r,\epsilon_p,\epsilon_y] = 2[q_2,q_3,q_4] = 2\left[E_x \sin\frac{\alpha}{2},\ E_y \sin\frac{\alpha}{2},\ E_z \sin\frac{\alpha}{2}\right] \approx [E_x\alpha,E_y\alpha,E_z\alpha] \quad (4)$$

Quaternion Combination for Successive Rotations

Consider a third coordinate system X',Y',Z' (system C) which is related to X,Y,Z by the direction cosine matrix $[C\leftarrow B]$. The matrix which relates X',Y',Z' to x,y,z is

$$[C\leftarrow A] = [C\leftarrow B][B\leftarrow A] \quad (5)$$

If $\bar{Q}_{BA} = [q_1,q_2,q_3,q_4]$ is the quaternion associated with $[B\leftarrow A]$ and $\bar{Q}_{CB} = [q'_1,q'_2,q'_3,q'_4]$ is the quaternion associated with $[C\leftarrow B]$, then the quaternion associated with $[C\leftarrow A]$, $\bar{Q}_{CA} = [q''_1,q''_2,q''_3,q''_4]$ was shown[2] by Euler to be

$$\begin{aligned} q''_1 &= q'_1q_1 - q'_2q_2 - q'_3q_3 - q'_4q_4 \\ q''_2 &= q'_1q_2 + q'_2q_1 + q'_3q_4 - q'_4q_3 \\ q''_3 &= q'_1q_3 - q'_2q_4 + q'_3q_1 + q'_4q_2 \\ q''_4 &= q'_1q_4 + q'_2q_3 - q'_3q_2 + q'_4q_1 \end{aligned} \quad (6)$$

This equation may be expressed in either of two matrix forms

$$\bar{Q}_{CA} = \begin{bmatrix} q''_1 \\ q''_2 \\ q''_3 \\ q''_4 \end{bmatrix} = \begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{bmatrix} \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \end{bmatrix} \quad (7)$$

or

$$\bar{Q}_{CA} = \begin{bmatrix} q''_1 \\ q''_2 \\ q''_3 \\ q''_4 \end{bmatrix} = \begin{bmatrix} q'_1 & -q'_2 & -q'_3 & -q'_4 \\ q'_2 & q'_1 & -q'_4 & q'_3 \\ q'_3 & q'_4 & q'_1 & -q'_2 \\ q'_4 & -q'_3 & q'_2 & q'_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (8)$$

The 4 × 4 matrix is defined‡ as the quaternion matrix $[M]$ and the minor of the first element is defined as the vector kernel $[V]$ of the matrix

‡ By the author.

$$[M] = \begin{bmatrix} q'_1 & -q'_2 & -q'_3 & -q'_4 \\ q'_2 & q'_1 & -q'_4 & q'_3 \\ q'_3 & q'_4 & q'_1 & -q'_2 \\ q'_4 & -q'_3 & q'_2 & q'_1 \end{bmatrix} \quad (9)$$

$$[V] = \begin{bmatrix} q'_1 & -q'_4 & q'_3 \\ q'_4 & q'_1 & -q'_2 \\ -q'_3 & q'_2 & q'_1 \end{bmatrix}$$

Using the quaternion matrix, the quaternion relating systems C and A may be written as

$$\bar{Q}_{CA} = [M(q')]\bar{Q}_{BA} \equiv [M_{CB}]\bar{Q}_{BA}$$

The matrix operator $[M_{CB}]$ is formed from the quaternion, $\bar{Q}_{CB}$, which defines the rotation from B to C. The argument of M indicates the quaternion from which the matrix was formed.

It may be seen by comparing Eqs. (7) and (8) that

$$\bar{Q}_{CA} = [M(q)]\dagger\bar{Q}_{CB} = [M_{BA}]\dagger\bar{Q}_{CB} \quad (10)$$

where $[M]\dagger$ is equal to $[M]$ with the vector kernel transposed

$$[M(u)]\dagger = \begin{bmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & u_4 & -u_3 \\ u_3 & -u_4 & u_1 & u_2 \\ u_4 & u_3 & -u_2 & u_1 \end{bmatrix}$$

and is called the quaternion *transmuted* matrix.§

Equations (9) and (10) demonstrate the rather remarkable result that the order of *quaternion multiplication*, as defined by Eqs. (6–8), is interchangeable even though the order of the corresponding matrix multiplication, in general, is not. Thus

$$\bar{Q}_{CA} = [M_{CB}]\bar{Q}_{BA} = [M_{BA}]\dagger\bar{Q}_{CB}$$

while for matrix multiplication

$$[C\leftarrow A] = [C\leftarrow B][B\leftarrow A] \neq [B\leftarrow A][C\leftarrow B] \neq [f(R)][C\leftarrow B]$$

where $[f(R)]$ is any rearrangement of the elements of $[B\leftarrow A]$. It is shown in Ref. 3 that this property may be extended to a product of three quaternions

$$\begin{aligned} \bar{Q}_{DA} &= [M_{DC}][M_{CB}]\bar{Q}_{BA} \\ &= [M_{DC}][M_{BA}]\dagger\bar{Q}_{CB} \\ &= [M_{BA}]\dagger[M_{DC}]\bar{Q}_{CB} \end{aligned}$$

and that in general

$$[M_{pq}][M_{BA}]\dagger = [M_{BA}]\dagger[M_{pq}]$$

where $\bar{Q}_{pq}$ and $\bar{Q}_{BA}$ are any two quaternions. The property of interchangeability may therefore be extended to the product of any number of quaternions. In any of the Eqs. (6, 7, 8, or 9) only 16 multiplications and 12 additions are required to completely define the quaternion $\bar{Q}_{CA}$. Multiplication of the two 3×3 direction cosine matrices $[C\leftarrow B][B\leftarrow A]$ to obtain $[C\leftarrow A]$ involves 27 multiplications. The extension to n successive rotations is obvious.

Application of the Quaternion Solution to Digital Control Systems

The result of the preceding development is that computations which require several successive coordinate system rotations may be simplified considerably by using quaternions instead of direction cosine matrices. Consider an inertially guided space vehicle which utilizes an airborne digital computer to perform both the guidance and the attitude control functions. The control system computations in such a vehicle will be performed much more rapidly (usually an order of magnitude) than the guidance computations. The control computations will be based upon 1) the commanded vehicle attitude with respect to an inertial computation frame, $[V_c\leftarrow I]$; 2) the transformation between the computation frame and inertially fixed platform axes, $[I\leftarrow P]$; 3) the transformation between the platform axes and a "gimbal frame" fixed to the outer case of the inertial measurement unit (IMU), $[P\leftarrow G]$; 4) the transformation between the "gimbal frame" and a vehicle frame whose axes are along the vehicle's roll, pitch, and yaw axes, $[G\leftarrow V]$.

In order to perform the control function, it is necessary to establish the actual vehicle attitude with respect to commanded attitude, $[V_c\leftarrow V]$. This can be accomplished by forming the product of the direction cosine matrices which represent the component rotations described previously;

$$[V_c\leftarrow V] = [V_c\leftarrow I][I\leftarrow P][P\leftarrow G][G\leftarrow V] \quad (11)$$

It has been shown previously that each of the rotations in Eq. (11) can be described by a quaternion. The equivalent quaternion relationship is

$$\bar{Q}_{V_cV} = [M_{V_cI}][M_{IP}][M_{PG}]\bar{Q}_{GV} \quad (12)$$

Only the IMU gimbal-to-platform rotation varies at the control computation frequency. The IMU gimbal system is fixed with respect to the vehicle, consequently, $\bar{Q}_{GV}$ is a constant. The commanded vehicle attitude varies at the guidance computation frequency and the platform-to-inertial rotation is primarily due to drift and changes slowly compared to the guidance computation frequency. Using the relationship of Eq. (10), Eq. (12) may be rewritten

$$\bar{Q}_{V_cV} = [M_{V_cI}][M_{IP}][M_{GV}]\dagger\bar{Q}_{PG} \quad (13)$$

The product $[M_{V_cI}][M_{IP}][M_{GV}]^{\neq}$ does not vary between guidance computations, therefore, the control computation is reduced to one quaternion multiplication;

$$\bar{Q}_{V_cV} = [M_{\text{guidance}}]\bar{Q}_{PG} \quad (14)$$

where $[M_{\text{guidance}}] = [M_{V_cI}][M_{IP}][M_{GV}]\dagger$ and is computed once each guidance cycle. The control computation is reduced from 54 multiplications¶ in Eq. (11) to 16 multiplications in Eq. (14). In addition, Eq. (4) has shown that the vector part of $\bar{Q}_{V_cV}$ yields the control error directly.

The percent computation reduction realized depends, of course, upon the data processing required to obtain $\bar{Q}_{PG}$. This is normally of the order of 50 multiplications or less. The over-all reduction in computation is, therefore, more than 40%.

Control Equations for a Three-Gimbal IMU

Consider, as an example, a digital control system performing Euler axis control on a vehicle whose attitude is measured by a three-gimbal IMU. The coordinate systems involved are 1) IMU platform coordinates P, assumed to be inertial; 2) IMU gimbal coordinates G, related to platform coordinates by the three gimbal angles θ_1, θ_2, θ_3; 3) Actual vehicle coordinates M; 4) Commanded vehicle coordinates M_c, which represent the desired vehicle attitude.

The attitude control system must perform the rotation from the M system to the M_c system and the control error is defined by the Euler axis and Euler angle which relate these two systems. The Euler axis and angle may be determined using Eqs. (2) if the direction cosine matrix relating M_c to M, $[M_c\leftarrow M]$ is computed. The direction cosine matrix $[M_c\leftarrow M]$ is computed by forming the product of the matrices

§ By the author.

¶ The product $[V_c\leftarrow I][I\leftarrow P]$ need only be computed once during the guidance cycle.

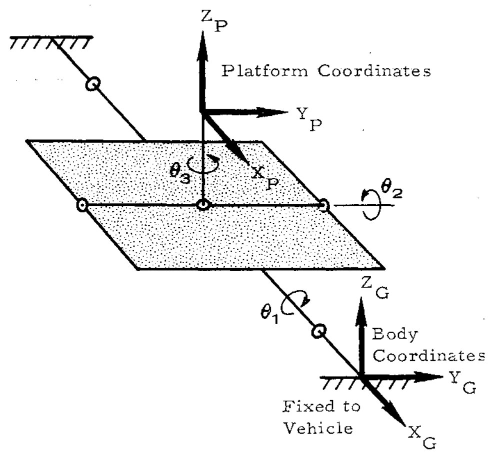

Fig. 3 Three-gimbal platform.

$[M_c \leftarrow P]$, $[P \leftarrow G]$, and $[G \leftarrow M]$;

$$[M_c \leftarrow M] = [M_c \leftarrow P][P \leftarrow G][G \leftarrow M] \tag{15}$$

The rotation $[M_c \leftarrow P]$ is the vehicle attitude command determined by guidance equations and $[P \leftarrow G]$ is a function of the IMU gimbal angles θ_1, θ_2, θ_3. The matrix $[G \leftarrow M]$ is a fixed constant which describes the mounting of the IMU with respect to the vehicle axes.

Of the three component rotations in Eq. (15), $[P \leftarrow G]$ will be changing most rapidly. Assume the control system computation frequency is ten times the guidance computation frequency. The matrix $[M_c \leftarrow P]$ is modified during each guidance cycle and therefore 10 control computations will be made using the same $[M_c \leftarrow P]$ matrix. The matrix $[M \leftarrow G]$ is a constant. Two matrix multiplications, however, must be performed at the control system computation rate since the order of matrix multiplication in Eq. (15) is not interchangeable.

If Eq. (15) is expressed in terms of quaternions,

$$\begin{aligned} \bar{Q}_{M_cM} &= [M_{M_cP}][M_{PG}]\bar{Q}_{GM} \\ &= [M_{M_cP}][M_{GM}]\dagger\bar{Q}_{PG} \end{aligned} \tag{16}$$

the product $[M_{M_cP}][M_{GM}]\dagger$ may be computed at the guidance rate and only one quaternion multiplication need be performed at the control system sampling frequency. Use of Eq. (15) requires 540 multiplications and 360 additions each guidance computation cycle whereas Eq. (16) requires only 176 multiplications and 132 additions, a reduction of more than 65% in computation time. There is also storage reduction associated with substituting Eq. (16) for Eq. (15) since the quaternion is composed of four elements and the direction cosine matrix requires nine.

The quaternion relating the IMU gimbals and platform, $\bar{Q}_{PG}$, is easily determined from the gimbal angles θ_1, θ_2, θ_3, since the axis of gimbal freedom in each case is known (and fixed). Figure 3 shows a typical three-gimbal configuration.

The matrix $[P \leftarrow G]$ may be determined by the product

$$[P \rightarrow G] = [\theta_3][\theta_2][\theta_1] \tag{17}$$

where

$$[\theta_3] = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\theta_2] = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$[\theta_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

The quaternion $\bar{Q}_{PG}$ is computed in an analogous manner;

$$\bar{Q}_{PG} = [M_{\theta_3}][M_{\theta_2}]\bar{Q}_{\theta_1} \tag{18}$$

where $[M_{\theta_2}]$ and $[M_{\theta_3}]$ are formed from $\bar{Q}_{\theta_2}$ and $\bar{Q}_{\theta_3}$, and

$$\begin{aligned} \bar{Q}_{\theta_3} &= [\cos(\theta_3/2),\ 0,\ 0,\ -\sin(\theta_3/2)] \\ \bar{Q}_{\theta_2} &= [\cos(\theta_2/2),\ 0,\ -\sin(\theta_2/2),\ 0] \\ \bar{Q}_{\theta_1} &= [\cos(\theta_1/2),\ -\sin(\theta_1/2),\ 0,\ 0] \end{aligned}$$

Expanded, $\bar{Q}_{PG}$ becomes

$$\bar{Q}_{PG} = \begin{bmatrix} C\theta_1/2\ C\theta_2/2\ C\theta_3/2 + S\theta_1/2\ S\theta_2/2\ S\theta_3/2 \\ S\theta_1/2\ C\theta_2/2\ C\theta_3/2 - C\theta_1/2\ S\theta_2/2\ S\theta_3/2 \\ C\theta_1/2\ S\theta_2/2\ C\theta_3/2 + S\theta_1/2\ C\theta_2/2\ S\theta_3/2 \\ C\theta_1/2\ C\theta_2/2\ S\theta_3/2 - S\theta_1/2\ S\theta_2/2\ C\theta_3/2 \end{bmatrix} \tag{19}$$

where C and S represent sine and cosine.

Equation (19) indicates that 12 multiplications and four additions are required to obtain $\bar{Q}_{PG}$ from the sines and cosines of the platform angles.** Fourteen multiplications and four additions are required to obtain the direction cosine matrix in Eq. (17). It is apparent, therefore, that the attitude quaternion is 1) simpler to obtain from platform angles and 2) easier to combine with guidance commands.

In addition, the quaternion formulation yields the control error directly by Eq. (4), whereas the direction cosine matrix must be manipulated, as in Eq. (2), to obtain the control error. Equally as important as the simplified form of Eqs. (18) and (19), however, is the fact that $\bar{Q}_{PG}$ may be expressed in terms of the angle which is changing most rapidly. For example, if θ_2 is changing much more rapidly than θ_1 or θ_3, the computation

$$\bar{Q}_{PG} = [M_{\theta_3}][M_{\theta_1}]\dagger\bar{Q}_{\theta_2}$$

allows part of the computation to be performed less frequently than the control cycle. Also, Eq. (18) lends itself to linear approximation by Taylor series expansion in terms of one or more of the gimbal angles.

Quaternion Expressions in Vector Form

Often it is simpler to work with and to visualize problems in vector rather than matrix form. Examination of Hamilton's definition of the quaternion (Ref. 1, Sec. IV) suggests that quaternion relationships might be readily expressed in vector form. In fact, inspection of Eq. (8) yields the following relationships:

$$q''_1 = q'_1 q_1 - [q'_2, q'_3, q'_4]\cdot[q_2, q_3, q_4] \tag{20}$$

$$[q''_2, q''_3, q''_4] = [q'_2, q'_3, q'_4] \times [q_2, q_3, q_4] + q'_1[q_2, q_3, q_4] + q_1[q'_2, q'_3, q'_4] \tag{21}$$

Consequently, if the quaternion $\bar{Q}$ is defined to consist of a scalar part $S = q_1$ and a vector part $\bar{V} = [q_2, q_3, q_4]$, then the relationship $Q'' = [M']Q$ from Eq. (6) may be expressed as

$$\begin{aligned} S'' &= S'S - \bar{V}'\cdot\bar{V} \\ \bar{V}'' &= \bar{V}' \times \bar{V} + S'\bar{V} + S\bar{V}' \end{aligned} \tag{22}$$

By interchanging the order of quaternion multiplication as indicated by Eq. (10) and expanding as in Eqs. (20) and (21), Eq. (22) may be expressed in terms of either S' and $\bar{V}'$ or S and $\bar{V}$;

$$\begin{aligned} S &= S''S' + \bar{V}''\cdot\bar{V}' \\ \bar{V} &= \bar{V}'' \times \bar{V}' + S'\bar{V}'' - S''\bar{V}' \end{aligned}$$

** Division of the platform angles by 2 is merely a shifting operation.

and

$$S' = S''S + \bar{V}''\cdot\bar{V}$$
$$\bar{V}' = \bar{V}\times\bar{V}'' + S\bar{V}'' - S''\bar{V}$$

Restrictions on the Use of Quaternions

There are no possible ambiguities in obtaining a resultant quaternion by multiplying or combining two quaternions. One and only one resultant arises from a quaternion multiplication and that resultant quaternion is the one associated with the matrix product due to the two component quaternions. If the rotation matrix $[A\leftarrow B]$ and $[C\leftarrow A]$ are represented by quaternions $\bar{Q}_{AB}$ and $\bar{Q}_{CA}$, then $\bar{Q}_{CB} = [M_{CA}]\bar{Q}_{AB}$ is unambiguous and $\bar{Q}_{CB}$ is the quaternion associated with $[C\leftarrow B] = [C\leftarrow A][A\leftarrow B]$. Furthermore, the attitude orientation described by any quaternion is unambiguous.

In Ref. 1, Sec. II, it is pointed out that certain difficulties may arise from the use of quaternions to describe attitude orientations. These apparent difficulties are readily circumvented by the adoption of certain conventions. The ambiguity arises if it becomes necessary to establish how the particular attitude was obtained or, conversely, how to undo the attitude orientation in the case of control system errors. Consider the quaternion

$$\bar{Q}_{AB} = \left[\cos\frac{w}{2}, X\sin\frac{w}{2}, Y\sin\frac{w}{2}, Z\sin\frac{w}{2}\right]$$

which describes the attitude of A with respect to B as a rotation w about the axis X,Y,Z. Notice that the same attitude is obtained by

a rotation:	about:
$-w$	$-X,-Y,-Z$
$w - 360°$	X,Y,Z
$360° - w$	$-X,-Y,-Z$

One quaternion, therefore, describes four attitudes and the corresponding control errors. Half of the four solutions can be eliminated by constraining $w \geq 0$. This leaves the two possibilities;

a rotation:	about:
w $(w \geq 0)$	X,Y,Z
$360° - w$ $(w \leq 360°)$	$-X,-Y,-Z$

Of these two possibilities one must involve a rotation of less than (or equal to) 180°. Therefore the "orientation ambiguity" can be eliminated by adopting the convention $0 \leq w \leq 180°$. This is the logical interpretation from the control standpoint since it is desirable to perform the smaller rotation $(w < 180°)$. For control purposes, the rotation of $-w$ about $-X,-Y,-Z$ would yield the same result as a rotation w about X,Y,Z. Consequently, no generality is lost by constraining $0 \leq w \leq 180°$. When $w = 180°$, the four possibilities reduce to two and either is valid from the control standpoint.

It is indicated in Ref. 1 that the error propagation due to a quaternion multiplication does not significantly exceed that due to direction cosine matrix multiplication. Although the results presented in Ref. 1 were based on analog simulation, the indication that quaternions are not considerably more sensitive to errors than direction cosine matrices is also valid for digital applications. Errors arise when one attempts to generate quaternions from direction cosine matrices which represent Euler rotations close to zero or 180°. This is due to the differencing of the small off-diagonal terms in the direction cosine matrix. This error source is present in the direction cosine control computations indicated by Eq. (2). Replacing cosine matrices with quaternions would, therefore, not increase the errors which arise due to near zero and 180° orientations and the use of a consistent medium (quaternions) could possibly reduce such errors by eliminating the need for processing the small off-diagonal terms of the direction cosine matrix.

The extent to which the magnitude of a quaternion varies from unity is a measure of the "nonorthogonality" of the associated direction cosine matrix.†† It is possible, therefore, to restore orthogonality to a quaternion whose elements have been perturbed by system errors simply by normalizing‡‡ the quaternion. It is very difficult, however, to adjust erroneous elements of a direction cosine matrix to satisfy orthogonality.

Conclusions

Use of quaternions to perform digital attitude computations where the desired end result is a control error vector results in a reduction of at least 40% in computation time over direction cosine matrix techniques. A reduction in storage is also realized due to the compact form of the quaternion. In addition, the quaternion equations yield the direction along which torque should be applied to perform an optimum rotation in the least angle sense. A form of "quaternion algebra" has been defined which allows the order of quaternion multiplication to be interchanged and which expresses quaternion relationships in compact matrix and vector forms suitable for digital programing The use of a quaternion algebra can simplify analysis of problems involving several coordinate system rotations as well as problems involving rotating vectors.

References

[1] Robinson, A. C., "On the Use of Quaternions in Simulation of Rigid-Body Motion," Dec. 1958, U.S. Air Force, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

[2] Whittaker, E. T., *Analytical Dynamics*, Dover, New York, 1944.

[3] Ickes, B. P., "Use of Quaternions to Perform Attitude and Control Computations," R:SEA-8029, Aug. 1967, Logicon Inc., San Pedro, Calif.

†† The direction cosine matrix orthogonality can be expressed as $\bar{R}_1 \times \bar{R}_2 = \bar{R}_3$ where $\bar{R}_1$ is a vector formed from the elements of the first row, $\bar{R}_2$ from the second row, and $\bar{R}_3$ from the third row. The same relationship applies to the matrix columns.

‡‡ Dividing by the square root of the sum of the squared quaternion elements.